

# A Conserved Cross Helicity for Non-Barotropic MHD

A. Yahalom\*

Ariel University, Ariel 40700, Israel

(Dated: May 10, 2016)

Cross helicity is not conserved in non-barotropic magnetohydrodynamics (MHD) (as opposed to barotropic or incompressible MHD). Here we show that variational analysis suggests a new kind of cross helicity which is conserved in the non barotropic case. The non barotropic cross helicity reduces to the standard cross helicity under barotropic assumptions. The new cross helicity is conserved even for topologies for which the variational principle does not apply.

PACS numbers: 03.65.Vf; 47.65.-d; 52.30.Cv

Keywords: Magnetohydrodynamics, Variational Analysis, Topological Conservation Laws

Cross Helicity was first described by Woltjer [1, 2] and is give by:

$$H_C \equiv \int \vec{B} \cdot \vec{v} d^3x, \quad (1)$$

in which  $\vec{B}$  is the magnetic field,  $\vec{v}$  is the velocity field and the integral is taken over the entire flow domain.  $H_C$  is conserved for barotropic or incompressible MHD and is given a topological interpretation in terms of the knottiness of magnetic and flow field lines. An analogous conserved helicity for fluid dynamics was obtained by Moffatt [3]. Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [4–7].

Consider the equations of non-barotropic MHD [8, 9]:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \quad (2)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (3)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (4)$$

$$\rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}, \quad (5)$$

$$\frac{dS}{dt} = 0. \quad (6)$$

In the above the following notations are utilized:  $\frac{\partial}{\partial t}$  is the temporal derivative,  $\frac{d}{dt}$  is the temporal material derivative and  $\vec{\nabla}$  has its standard meaning in vector calculus.  $\rho$  is the fluid density and  $S$  is the specific entropy. Finally  $p(\rho, S)$  is the pressure which depends on the density

and entropy (the non-barotropic case). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (3) describes the fact that the magnetic field is solenoidal, equation (4) describes the conservation of mass and equation (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight  $(\vec{v}, \vec{B}, \rho, S)$  and the number of equations (2,4,5,6) is also eight. Notice that equation (3) is a condition on the initial  $\vec{B}$  field and is satisfied automatically for any other time due to equation (2).

In non-barotropic MHD one can calculate the temporal derivative of the cross helicity (1) using the above equations and obtain:

$$\frac{dH_C}{dt} = \int T \vec{\nabla} S \cdot \vec{B} d^3x, \quad (7)$$

in which  $T$  is the temperature. Hence, generally speaking cross helicity is not conserved.

A clue on how to define cross helicity for nonbarotropic MHD can be obtained from the variational analysis described in [9] which is valid for magnetic field lines at the intersection of two comoving surfaces  $\chi, \eta$  (Euler potentials). Following Sakurai [10] the magnetic field takes the form:

$$\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta. \quad (8)$$

In terms of the functions  $\chi_i \equiv (\chi, \eta, S)$  one obtains the five function Lagrangian density [9]:

$$\begin{aligned} \hat{\mathcal{L}}[\chi_i, \nu, \rho] = & \rho \left[ \frac{1}{2} A_{jn}^{-1} \frac{\partial \chi_j}{\partial t} \frac{\partial \chi_n}{\partial t} + \frac{\partial \nu}{\partial \chi_m} \frac{\partial \chi_m}{\partial t} \right. \\ & \left. - \frac{\partial \nu}{\partial t} - \varepsilon(\rho, \chi_3) \right] - \frac{1}{8\pi} (\vec{\nabla} \chi_1 \times \vec{\nabla} \chi_2)^2. \end{aligned} \quad (9)$$

The two additional functions on which the Lagrangian density depend are a Bernoulli type function  $\nu$  and the mass density  $\rho$ . The Lagrangian density depend on the  $\chi_i$  fields through the inverse of the symmetric  $A$  matrix

\* asya@ariel.ac.il

defined as:

$$A_{ij} \equiv \vec{\nabla} \chi_i \cdot \vec{\nabla} \chi_j \quad (10)$$

and through the specific internal energy  $\varepsilon$  which is a function of density and entropy. Einstein summation convention is assumed throughout.

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [8] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [11] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field  $\vec{B}$  the velocity field  $\vec{v}$  and the pressure  $P$ . Kats [12] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Sakurai [10] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [8]. Yahalom & Lynden-Bell [7] combined the Lagrangian of Sturrock [8] with the Lagrangian of Sakurai [10] to obtain an **Eulerian** Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [13] (see also [14]). Yahalom [18] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [19] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [15] in terms of 15 functions and V.A. Kats [12] in terms of 20 functions. Morrison [16] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [16]). The variational principle introduced in [9] show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [10] representation for the magnetic field.

The variational equations are given in terms of the quantities:

$$\alpha_i \equiv (\alpha, \beta, \sigma), \quad \alpha_i[\chi_i, \nu] = -A_{ij}^{-1} \left( \frac{\partial \chi_j}{\partial t} + \vec{\nabla} \nu \cdot \vec{\nabla} \chi_j \right). \quad (11)$$

And the generalized Clebsch representation of the velocity [9]:

$$\vec{v} = \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta + \sigma \vec{\nabla} S. \quad (12)$$

as follows:

$$\frac{d\nu}{dt} = \frac{1}{2} \vec{v}^2 - w, \quad (13)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (14)$$

$$\frac{d\sigma}{dt} = T, \quad (15)$$

$$\frac{d\alpha}{dt} = \frac{\vec{\nabla} \eta \cdot \vec{J}}{\rho}, \quad (16)$$

$$\frac{d\beta}{dt} = -\frac{\vec{\nabla} \chi \cdot \vec{J}}{\rho}. \quad (17)$$

In the above:  $w$  is the specific enthalpy and the current is  $\vec{J} = \frac{\vec{\nabla} \times \vec{B}}{4\pi}$ . The above equations are shown [9] to be equivalent to the non barotropic MHD equations (2-6).

The function  $\nu$  whose material derivative is given in equation (13) can be multiple valued as only its gradient appears in the velocity (12). However, the discontinuity of  $\nu$  is a conserved quantity :

$$\frac{d[\nu]}{dt} = 0. \quad (18)$$

since the right hand side of equation (13) are physical and hence single valued quantities. A similar equation hold also for barotropic fluid dynamics and barotropic MHD [7, 17-19].

Let us now write the cross helicity given in equation (1) in terms of equation (8) and equation (12), this will take the form:

$$H_C = \int d\Phi [\nu] + \int d\Phi \oint \sigma dS \quad (19)$$

in which:  $d\Phi = \vec{B} \cdot d\vec{A} = \vec{\nabla} \chi \times \vec{\nabla} \eta \cdot d\vec{A} = d\chi d\eta$  and the closed line integral is taken along a magnetic field line.  $d\Phi$  is a magnetic flux element which is comoving according to equation (2) and  $d\vec{A}$  is an infinitesimal area element. Although the cross helicity is not conserved for non-barotropic flows, looking at the right hand side we see that it is made of a sum of two terms. One which is conserved as both  $d\Phi$  and  $[\nu]$  are comoving and one which is not. This suggests the following definition for the non barotropic cross helicity  $H_{CNB}$ :

$$H_{CNB} \equiv \int d\Phi [\nu] = H_C - \int d\Phi \oint \sigma dS \quad (20)$$

Which can be written in a more conventional form:

$$H_{CNB} = \int \vec{B} \cdot \vec{v}_t d^3x \quad (21)$$

in which the topological velocity field is defined as follows:

$$\vec{v}_t = \vec{v} - \sigma \vec{\nabla} S \quad (22)$$

It should be noticed that  $H_{CNB}$  is conserved even for an MHD not satisfying the Sakurai topological constraint

given in equation (8), provided that we have a field  $\sigma$  satisfying the equation  $\frac{d\sigma}{dt} = T$ . Thus the non barotropic cross helicity conservation law:

$$\frac{dH_{CNB}}{dt} = 0, \quad (23)$$

is more general than the variational principle described by equation (9) as follows from a direct computation using equations (2,4,5,6). Also notice that for a constant specific entropy  $S$  we obtain  $H_{CNB} = H_C$  and the non-barotropic cross helicity reduces to the standard

barotropic cross helicity. To conclude we introduce also a local topological conservation law in the spirit of [19] which is the non barotropic cross helicity per unit of magnetic flux. This quantity which is equal to the discontinuity of  $\nu$  is conserved and can be written as a sum of the barotropic cross helicity per unit flux and the closed line integral of  $Sd\sigma$  along a magnetic field line:

$$[\nu] = \frac{dH_{CNB}}{d\Phi} = \frac{dH_C}{d\Phi} + \oint Sd\sigma. \quad (24)$$

- 
- [1] Woltjer L, . 1958a Proc. Nat. Acad. Sci. U.S.A. 44, 489-491.
  - [2] Woltjer L, . 1958b Proc. Nat. Acad. Sci. U.S.A. 44, 833-841.
  - [3] Moffatt H. K. J. Fluid Mech. 35 117 (1969)
  - [4] A. Yahalom, "Helicity Conservation via the Noether Theorem" J. Math. Phys. 36, 1324-1327 (1995). [Los-Alamos Archives solv-int/9407001]
  - [5] N. Padhye and P. J. Morrison, Phys. Lett. A 219, 287 (1996).
  - [6] N. Padhye and P. J. Morrison, Plasma Phys. Rep. 22, 869 (1996).
  - [7] Yahalom A. and Lynden-Bell D., "Simplified Variational Principles for Barotropic Magnetohydrodynamics," (Los-Alamos Archives- physics/0603128) *Journal of Fluid Mechanics*, Vol. 607, 235-265, 2008.
  - [8] P. A. Sturrock, *Plasma Physics* (Cambridge University Press, Cambridge, 1994)
  - [9] A. Yahalom "Simplified Variational Principles for non Barotropic Magnetohydrodynamics". (arXiv: 1510.00637 [Plasma Physics]) J. Plasma Phys. (2016), vol. 82, - 905820204 doi:10.1017/S0022377816000222.
  - [10] Sakurai T., "A New Approach to the Force-Free Field and Its Application to the Magnetic Field of Solar Active Regions," Pub. Ast. Soc. Japan, Vol. 31, 209, 1979.
  - [11] V. A. Vladimirov and H. K. Moffatt, J. Fluid. Mech. **283** 125-139 (1995)
  - [12] A. V. Kats, Los Alamos Archives physics-0212023 (2002), JETP Lett. 77, 657 (2003)
  - [13] A. Frenkel, E. Levich and L. Stilman Phys. Lett. A **88**, p. 461 (1982)
  - [14] V. E. Zakharov and E. A. Kuznetsov, Usp. Fiz. Nauk 40, 1087 (1997)
  - [15] J. D. Bekenstein and A. Oron, Physical Review E Volume 62, Number 4, 5594-5602 (2000)
  - [16] P.J. Morrison, Poisson Brackets for Fluids and Plasmas, AIP Conference proceedings, Vol. 88, Table 2 (1982).
  - [17] Asher Yahalom and Donald Lynden-Bell "Variational Principles for Topological Barotropic Fluid Dynamics" Geophysical & Astrophysical Fluid Dynamics. 11/2014; 108(6). DOI: 10.1080/03091929.2014.952725.
  - [18] Yahalom A., "A Four Function Variational Principle for Barotropic Magnetohydrodynamics" EPL 89 (2010) 34005, doi: 10.1209/0295-5075/89/34005 [Los - Alamos Archives - arXiv: 0811.2309]
  - [19] Asher Yahalom "Aharonov - Bohm Effects in Magnetohydrodynamics" Physics Letters A. Volume 377, Issues 31-33, 30 October 2013, Pages 1898-1904.